

# HOLOMORPHIC FACTORIZATION OF DETERMINANTS OF LAPLACIANS USING QUASI-FUCHSIAN UNIFORMIZATION

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**ABSTRACT.** For a quasi-Fuchsian group  $\Gamma$  with ordinary set  $\Omega$ , and  $\Delta_n$  the Laplacian on  $n$ -differentials on  $\Gamma \backslash \Omega$ , we define a notion of a Bers dual basis  $\phi_1, \dots, \phi_{2d}$  for  $\ker \Delta_n$ . We prove that  $\det \Delta_n / \det \langle \phi_j, \phi_k \rangle$ , is, up to an anomaly computed by Takhtajan and the second author in [TT03a], the modulus squared of a holomorphic function  $F(n)$ , where  $F(n)$  is a quasi-Fuchsian analogue of the Selberg zeta  $Z(n)$ . This generalizes the D'Hoker-Phong formula  $\det \Delta_n = c_{g,n} Z(n)$ , and is a quasi-Fuchsian counterpart of the result for Schottky groups proved by Takhtajan and the first author in [MT04].

## 1. INTRODUCTION

Let  $\Gamma$  be a Fuchsian group, such that  $\Gamma \backslash \mathbb{H}_+ \simeq X$  is a compact Riemann surface of genus  $g > 1$ , let  $X$  be endowed with the hyperbolic metric, and let  $\Delta_n$  be the Laplacian acting on the  $n$ th power of the canonical bundle of  $X$ ,  $n > 1$ . (See Section 2 for definitions.) In [DP86], D'Hoker and Phong computed the zeta-regularized determinant of  $\det \Delta_n$  in terms of the group  $\Gamma$ :

$$(1.1) \quad \det \Delta_n = c_{g,n} Z(n) = c_{g,n} \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - \lambda(\gamma)^{n+m}),$$

where  $\{\gamma\}$  runs over primitive conjugacy classes in  $\Gamma$ ,  $\lambda(\gamma)$  is the multiplier of  $\gamma$ , and  $c_{g,n}$  is a known constant. This theorem is an analogue of the expression, (essentially due to Kronecker), when  $X$  has genus 1 and a flat metric of area 1,

$$(1.2) \quad \det' \Delta_0(\tau) = 4 \operatorname{Im} \tau |\eta(\tau)|^4,$$

where  $X = \langle z \mapsto z + 1, z \mapsto z + \tau \rangle \backslash \mathbb{C}$ , and  $\eta$  is the Dedekind function,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m), \quad q = e^{2\pi i \tau}.$$

The expression (1.1) is a higher genus analogue of (1.2), but other analogues — in some sense closer to (1.2) — are possible.

In [ZT87a], Takhtajan and Zograf proved the following formula on the Teichmüller space  $T_g$ :

$$(1.3) \quad \partial_\mu \partial_{\bar{\nu}} \log \frac{\det \Delta_n}{\det N_n} = \frac{6n^2 - 6n + 1}{12\pi} \langle \mu, \nu \rangle_{WP}.$$

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Here  $\mu$  and  $\nu$  are holomorphic tangent vectors,  $(N_n)_{jk} = \langle \phi_j, \phi_k \rangle$  for a holomorphically varying basis  $\phi_1, \dots, \phi_d$  of holomorphic  $n$ -differentials,  $(d = (2n - 1)(g - 1))$ , which we call the period matrix, and  $\langle \cdot, \cdot \rangle_{WP}$  is the Weil-Petersson metric on  $T_g$ .

Now, for each point  $[X] \in T_g$ , there is a unique normalized marked Schottky group  $\Gamma$  with ordinary set  $\Omega$ , such that  $X \simeq \Gamma \backslash \Omega$ . As a corollary to their construction of the Liouville action functional for Schottky groups, Takhtajan and Zograf constructed the classical Liouville action [ZT87b], a positive real-valued function  $S$  on the Schottky space (whose points correspond to marked Schottky groups), and hence on the Teichmüller space  $T_g$  as well. The negative of this function is a potential for the Weil-Petersson metric, namely,

$$\partial_\mu \partial_{\bar{\nu}}(-S) = \langle \mu, \nu \rangle_{WP}.$$

Therefore,

$$\log \frac{\det \Delta_n}{\det N_n} + \frac{6n^2 - 6n + 1}{12\pi} S$$

is a pluriharmonic real-valued function on  $T_g$ . Since  $T_g$  is contractible, there exists a holomorphic function  $F(n)$  on  $T_g$  such that

$$\log \frac{\det \Delta_n}{\det N_n} + \frac{6n^2 - 6n + 1}{12\pi} S = \log |F(n)|^2,$$

or equivalently,

$$\frac{\det \Delta_n}{\det N_n} = \exp \left( -\frac{6n^2 - 6n + 1}{12\pi} S \right) |F(n)|^2,$$

which is what we refer to as “holomorphic factorization”.

In [MT04], L. Takhtajan and the first author showed that, with a suitable explicit choice of the basis  $\phi_1, \dots, \phi_d$ , depending on the Schottky group  $\Gamma$ , the function  $F(n)$  has the expression

$$F(n) = (1 - \lambda(L_1))^2 (1 - \lambda(L_1)^2)^2 \cdots (1 - \lambda(L_1)^{n-1})^2 (1 - \lambda(L_2)^{n-1}) \times \\ \cdot \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - \lambda(\gamma)^{n+m}),$$

where  $\{\gamma\}$  runs over primitive conjugacy classes in the Schottky group  $\Gamma$ , and  $L_1, \dots, L_g$  are generators of  $\Gamma$  corresponding to the marking. (The factors before the product sign are related to the explicit choice of basis  $\phi_1, \dots, \phi_d$ .) This theorem is closer in form to the genus 1 result than is (1.1), and in fact specializes to it when  $g = 1$ .

In [TT03a], L. Takhtajan and the second author extended the results of [ZT87b] to quasi-Fuchsian groups. In particular, they constructed the classical Liouville action, a positive real-valued function  $S$  on the quasi-Fuchsian space  $QF_g$  (whose points correspond to marked quasi-Fuchsian groups). We have  $QF_g \simeq T_g \times T_g$ . As in the Schottky case, the negative of this function is a potential for the Weil-Petersson metric. Therefore one may expect that a result similar to [MT04] will hold for quasi-Fuchsian groups. In this paper we prove such a result.

First it is necessary to make an appropriate choice of basis for holomorphic  $n$ -differentials. For this purpose we define the notion of a *Bers dual* basis. Let  $\Gamma$  be a cocompact quasi-Fuchsian group with domain of discontinuity  $\Omega = \Omega_+ \sqcup \Omega_-$ , simultaneously uniformizing the compact Riemann surfaces  $X_\pm \simeq \Gamma \backslash \Omega_\pm$ . For

$n > 1$ , Bers [Ber66] introduced an invertible integral operator  $K_-$  which maps the conjugate of a holomorphic  $n$ -differential on  $X_-$  to a holomorphic  $n$ -differential on  $X_+$ . Its kernel is given by

$$K_-(z, w) = \frac{2^{2n-2}(2n-1)}{\pi} \sum_{\gamma \in \Gamma} \frac{\gamma'(z)^n}{(\gamma z - w)^{2n}}, \quad z \in \Omega_+, w \in \Omega_-.$$

Given a basis  $\phi_1^+, \dots, \phi_d^+$  of holomorphic  $n$ -differentials of  $X_+ \simeq \Gamma \backslash \Omega_+$ , the Bers dual is the basis  $\phi_1^-, \dots, \phi_d^-$  of holomorphic  $n$ -differentials of  $X_- \simeq \Gamma \backslash \Omega_-$  such that

$$K_-(z, w) = \sum_{k=1}^d \phi_k^+(z) \phi_k^-(w).$$

Now, the Local Index Theorem (1.3) and the result in [TT03a] imply that on the quasi-Fuchsian space, there exists a holomorphic function  $F(n)$  such that

$$\frac{\det \Delta_n(X_+)}{\det N_n([X_+])} \frac{\det \Delta_n(X_-)}{\det N_n([X_-])} = \exp \left( -\frac{6n^2 - 6n + 1}{12\pi} S \right) |F(n)|^2.$$

Here  $(N([X_\pm]))_{jk} = \langle \phi_j^\pm, \phi_k^\pm \rangle$  for Bers dual bases  $\phi_1^+, \dots, \phi_d^+$  and  $\phi_1^-, \dots, \phi_d^-$ . In this paper, we show that up to a known multiplicative constant, the function  $F(n)$  is given by the product

$$F(n) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - \lambda(\gamma)^{n+m}),$$

where  $\{\gamma\}$  runs over the set of conjugacy classes of primitive elements of the quasi-Fuchsian group  $\Gamma$ .

This theorem specializes to the D'Hoker-Phong result when restricted to the subspace of Fuchsian groups. In fact, the method of proof is to use (1.1), together with pluriharmonicity (coming from the Local Index Theorem), and symmetry properties of the various quantities in the theorem under complex conjugation of the group  $\Gamma \mapsto \bar{\Gamma}$ .

We collect the background facts we will need in Section 2. In following sections, we discuss the generalization of the Local Index Theorem, the function  $F(n)$ , the Bers integral operator and Bers dual basis. The main theorem is stated and proved in Section 6.

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## 2. PRELIMINARIES

In this section, we collect some necessary facts and definitions. We refer the reader to the references cited for further details.

**2.1. Quasi-Fuchsian groups and simultaneous uniformization.** (See [Ahl87, Ber70, Ber71, Ber81, Kra72, TT03a].) By definition, a *Kleinian group* is a discrete subgroup  $\Gamma$  of the group of Möbius transformations  $\text{PSL}(2, \mathbb{C})$  which acts properly discontinuously on some non-empty open subset of the Riemann sphere  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . The largest such subset  $\Omega \subset \hat{\mathbb{C}}$  is called the *ordinary set* of  $\Gamma$  and its complement  $\Lambda$  is called the *limit set* of  $\Gamma$ .

A Kleinian group  $\Gamma$  is called a *quasi-Fuchsian group* if it leaves some directed Jordan curve  $\mathcal{C} \in \widehat{\mathbb{C}}$  invariant. We have  $\Lambda \subset \mathcal{C}$ ; if  $\Lambda = \mathcal{C}$ , the group is said to be *of the first kind*. In this case,  $\Omega$  consists of the two domains  $\Omega_+$  and  $\Omega_-$  complementary to  $\mathcal{C}$ , chosen such that the boundary of  $\Omega_{\pm}$  (with the orientation from the complex plane) is  $\pm\mathcal{C}$ . If  $\mathcal{C}$  is a circle or line, the group is called *Fuchsian*; in this case, we will assume  $\mathcal{C}$  has been conjugated to  $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  with the usual orientation, so  $\Omega_{\pm} = \mathbb{H}_{\pm}$ , the upper (resp. lower) half plane.

It is known that a group  $\Gamma$  is quasi-Fuchsian if and only if it is the deformation of a Fuchsian group under a quasiconformal (q. c.) map, that is,

$$\Gamma = w\Gamma_0w^{-1}$$

for some Fuchsian group  $\Gamma_0$  and some q. c. homeomorphism  $w : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ . (See the references for the definition of a q. c. map.)

An element  $\gamma$  of  $\mathrm{PSL}(2, \mathbb{C})$  is called *loxodromic* if it is conjugate in  $\mathrm{PSL}(2, \mathbb{C})$  to  $z \mapsto \lambda(\gamma)z$  for some  $\lambda(\gamma) \in \mathbb{C}$  (called the *multiplier*) such that  $0 < |\lambda(\gamma)| < 1$ ; if every element of a Kleinian group is loxodromic except the identity, the group is called *totally loxodromic*. If  $\Gamma$  is a finitely generated, totally loxodromic quasi-Fuchsian group of the first kind, then the quotient  $\Gamma \backslash \Omega \simeq X$  has two connected components  $\Gamma \backslash \Omega_{\pm} \simeq X_{\pm}$ , which are nonsingular compact Riemann surfaces of common genus  $g > 1$ . If  $\Gamma$  is Fuchsian as well, then  $X_- \simeq \overline{X_+}$ , the mirror image of  $X_+$  (obtained by taking the complex conjugate of all local coordinates). Conversely, Bers simultaneous uniformization theorem states that, given any two compact Riemann surfaces  $X_+$ ,  $X_-$  of common genus  $g > 1$ , there exists a finitely generated totally loxodromic quasi-Fuchsian group of the first kind  $\Gamma$  such that  $\Gamma \backslash \Omega_{\pm} \simeq X_{\pm}$ .

*In the sequel, “quasi-Fuchsian group” will always refer to a finitely generated, totally loxodromic quasi-Fuchsian group of the first kind.*

A *marked* quasi-Fuchsian group is a quasi-Fuchsian group  $\Gamma$  (with the above convention), together with a choice  $a_1, \dots, a_g, b_1, \dots, b_g \in \Gamma$  of generators corresponding to standard generators of  $\pi_1(X_+)$ . The marking directs  $\mathcal{C}$  by taking the segment from the attracting fixed point of  $a_1$  to that of  $a_2$  to be positively directed. A marked quasi-Fuchsian group is *normalized* if the attracting fixed points of  $a_1$  and  $a_2$  are 0 and 1 respectively, and the repelling fixed point of  $a_1$  is  $\infty$ . A marked quasi-Fuchsian group may be normalized by a (unique) overall conjugation in  $\mathrm{PSL}(2, \mathbb{C})$ .

**2.2.  $n$ -differentials and determinants of Laplacians.** (See [TZ91] and references therein.) Suppose that  $\Gamma$  is a quasi-Fuchsian group, with  $\Gamma \backslash \Omega_{\pm} \simeq X_{\pm}$  compact Riemann surfaces of genus  $g > 1$ . For integers  $n$  and  $m$ , an *automorphic form of type  $(n, m)$*  is a function  $\phi : \Omega_{\pm} \rightarrow \widehat{\mathbb{C}}$  such that

$$\phi(z) = \phi(\gamma z) \gamma'(z)^n \overline{\gamma'(z)}^m \quad \text{for all } z \in \Omega_{\pm}, \gamma \in \Gamma.$$

We write  $B_{n,m}(\Omega_{\pm}, \Gamma)$ , for the space of smooth automorphic forms of type  $(n, m)$ , and identify  $B_{n,m}(\Omega_{\pm}, \Gamma)$  with  $B_{n,m}(X_{\pm})$ , the space of smooth sections of  $\omega_{X_{\pm}}^n \otimes \overline{\omega}_{X_{\pm}}^m$ , where  $\omega_{X_{\pm}} = T^*X_{\pm}$  is the holomorphic cotangent bundle of  $X_{\pm}$ . We abbreviate  $B_{n,0}(\Omega_{\pm}, \Gamma) = B_n(\Omega_{\pm}, \Gamma)$ , called  *$n$ -differentials*, and write  $A_n(\Omega_{\pm}, \Gamma)$  for the holomorphic  $n$ -differentials.

The hyperbolic metric on  $X_\pm$ , written locally as  $\rho(z) |dz|^2$ , induces a Hermitian metric

$$(2.1) \quad \langle \phi, \psi \rangle = \int_{\mathcal{F}} \phi \bar{\psi} \rho^{1-n-m} d^2 z,$$

on  $B_{n,m}(X_\pm)$ , where  $\mathcal{F}_\pm$  is a fundamental region for  $\Gamma$  in  $\Omega_\pm$ , and  $d^2 z = \frac{i}{2} dz \wedge d\bar{z}$  is the Euclidean area form on  $\Omega_\pm$ . The complex structure and metric determine a connection

$$D = \partial_n \oplus \bar{\partial}_n : B_n(X_\pm) \rightarrow B_{n+1}(X_\pm) \oplus B_{n,1}(X_\pm)$$

on the line bundle  $\omega_{X_\pm}^n$ , given locally by

$$\bar{\partial}_n = \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \partial_n = \rho^n \frac{\partial}{\partial z} \rho^{-n}.$$

The  $\bar{\partial}$ -Laplacian acting on  $B_n(X_\pm)$  is then  $\Delta_n = \bar{\partial}_n^* \bar{\partial}_n$ , where  $\bar{\partial}_n^* = -\rho^{-1} \partial_n$  is the adjoint of  $\bar{\partial}_n$  with respect to (2.1).

The operator  $\Delta_n$  is self-adjoint and non-negative, and has pure discrete spectrum in the  $L^2$ -closure of  $B_{n,m}(X_\pm)$ . The corresponding eigenvalues  $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$  of  $\Delta_n$  have finite multiplicity and accumulate only at infinity. The determinant of  $\Delta_n$  is defined by zeta regularization: the elliptic operator zeta-function

$$\zeta_n(s) = \sum_{\lambda_k > 0} \lambda_k^{-s},$$

defined initially for  $\text{Re } s > 1$ , has a meromorphic continuation to the entire  $s$ -plane, and by definition

$$\det \Delta_n = e^{-\zeta_n'(0)}.$$

The non-zero spectrum of  $\Delta_{1-n}$  is identical to that of  $\Delta_n$ , so that  $\det \Delta_n = \det \Delta_{1-n}$ . Hence without loss of generality we will usually assume  $n \geq 1$ .

For  $n \geq 2$ ,  $\ker \Delta_n = A_n(X_\pm)$  has dimension  $d = (2n-1)(g-1)$ . If  $\phi_1, \dots, \phi_d$  is a basis for  $A_n(X_\pm)$ , we refer to  $(N_n)_{jk} = \langle \phi_j, \phi_k \rangle$  as the *period matrix* corresponding to this basis.

**2.3. Quasi-Fuchsian deformation space.** (See [Ahl87, Ber81, TT03a], and references therein.) The set of marked, normalized quasi-Fuchsian groups of genus  $g > 1$  (recall our conventions on quasi-Fuchsian groups from Section 2.1) has a natural structure of a complex manifold of dimension  $6g - 6$ . We refer to it as the *quasi-Fuchsian space of genus  $g$*  and denote it by  $QF_g$ . The subset of  $QF_g$  corresponding to Fuchsian groups — that is, the subset with  $X_- \simeq \overline{X_+}$  — is called the Teichmüller space,  $T_g$ . It is a totally real submanifold of  $QF_g$ ; however, it has a natural complex structure. With this complex structure, there is a natural biholomorphism [Kra72]

$$(2.2) \quad QF_g = T_g \times T_g.$$

We obtain the first (resp. second) copy of  $T_g$  by fixing  $X_-$  (resp.  $X_+$ ) (this is the *Bers embedding* of  $T_g$  into  $QF_g$ ).

Local coordinates (*Bers coordinates*) for  $QF_g$  may be defined as follows. Fix a quasi-Fuchsian group  $\Gamma$  of genus  $g$ , which will serve as the basepoint for the coordinate chart. Let  $\Omega_{-1,1}(\Omega, \Gamma)$  be the space of *harmonic Beltrami differentials* for  $\Gamma$ , that is,  $\mu \in B_{-1,1}(\Omega, \Gamma)$  such that  $(\rho\mu)_z = 0$ . It is a complex vector space of dimension  $6g - 6$ ; we give it the sup norm. Each  $\mu$  in the open unit ball in

$\Omega_{-1,1}(\Omega, \Gamma)$  is identified with a group  $\Gamma^\mu$ , a deformation of  $\Gamma$ , by the generalized Riemann mapping theorem: extending  $\mu$  to  $\hat{\mathbb{C}}$  by  $\mu|_\Lambda = 0$ , there exists a unique q. c. homeomorphism  $w_\mu : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  satisfying the Beltrami equation

$$(2.3) \quad (w_\mu)_{\bar{z}} = \mu(w_\mu)_z$$

and fixing the points 0, 1 and  $\infty$ . Set

$$\Gamma^\mu = w_\mu \Gamma w_\mu^{-1}.$$

The group  $\Gamma^\mu$  is quasi-Fuchsian, marked, and normalized, so it may be identified with a point in  $QF_g$ ; for a sufficiently small neighbourhood of the origin in  $\Omega_{-1,1}(\Omega, \Gamma)$ , this identification is injective.

Write  $U(\Gamma)$  for the Bers coordinate chart based at  $\Gamma$ . There is a natural bi-holomorphism  $U(\Gamma) \simeq U(\Gamma^\mu)$ , mapping  $\Gamma^\nu \in U(\Gamma)$  to  $(\Gamma^\mu)^\lambda \in U(\Gamma^\mu)$  such that  $w_\nu = w_\lambda \circ w_\mu$ , which provides overlap maps for the coordinate charts, and allows us to identify the holomorphic tangent space at the point  $\Gamma^\nu \in U(\Gamma)$  with the space of harmonic Beltrami differentials  $\Omega_{-1,1}(\Omega^\nu, \Gamma^\nu)$ . Given  $\mu \in \Omega_{-1,1}(\Omega, \Gamma)$ , we denote by  $\partial_\mu$  and  $\partial_{\bar{\mu}}$  the holomorphic and anti-holomorphic derivatives (vector fields) in a neighbourhood of  $\Gamma$  defined using the Bers coordinates at the point  $\Gamma$ . The scalar product (2.1) on  $\Omega_{-1,1}(\Gamma^\nu)$  defines a Kähler metric on  $QF_g$  — the *Weil-Petersson metric*.

To cover  $QF_g$ , it is sufficient to take Bers coordinate charts based at Fuchsian groups. Explicitly: given a Fuchsian group  $\Gamma_0$ , let  $\mu_1, \dots, \mu_{6g-6}$  be a real basis of  $\Omega_{-1,1}(\Gamma_0)$  which satisfies

$$\mu(\bar{z}) = \overline{\mu(z)},$$

and map  $\epsilon = (\epsilon_1, \dots, \epsilon_{6g-6}) \mapsto \Gamma^\epsilon = w_\epsilon \circ \Gamma_0 \circ w_\epsilon^{-1}$ , where  $w_\epsilon : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is the unique normalized q. c. mapping with Beltrami differential  $\epsilon_1 \mu_1 + \dots + \epsilon_{6g-6} \mu_{6g-6}$ . In this coordinate chart, the subset  $\text{Im } \epsilon_k = 0$ ,  $k = 1, \dots, 6g-6$  consists of precisely the Fuchsian groups in the chart. These coordinate charts cover  $QF_g$ .

For  $\Gamma$  quasi-Fuchsian and  $\mu$  in the unit ball in  $\Omega_{-1,1}(\Gamma)$ , if  $w_\epsilon$  is the corresponding q. c. map satisfying (2.3), we then have, for each  $z \in \Omega$ ,

$$(2.4) \quad \left. \frac{\partial}{\partial \bar{\epsilon}} \right|_{\epsilon=0} w_{\epsilon\mu}(z) = 0.$$

Combining this with the equation

$$(2.5) \quad \gamma^{\epsilon\mu} \circ w_{\epsilon\mu} = w_{\epsilon\mu} \circ \gamma,$$

we have

$$(2.6) \quad \left. \frac{\partial}{\partial \bar{\epsilon}} \right|_{\epsilon=0} \gamma^{\epsilon\mu} = 0$$

for each  $\gamma^{\epsilon\mu} \in \Gamma^{\epsilon\mu}$ .

**2.4. Families of  $n$ -differentials.** (See [Ber81] and references therein.) The quasi-Fuchsian fibre space is a fibration  $p : \mathcal{ZF}_g \rightarrow QF_g$  with fibre  $\pi^{-1}(t) = \Gamma^t \backslash \Omega^t$  for  $t \in QF_g$ . Let  $T_V \mathcal{ZF}_g \rightarrow \mathcal{ZF}_g$  be the holomorphic vertical tangent bundle — the holomorphic line bundle over  $\mathcal{ZF}_g$  consisting of vectors in the holomorphic tangent space  $T\mathcal{ZF}_g$  that are tangent to the fibres  $\pi^{-1}(t)$ . A family  $\phi^t$  of  $n$ -differentials is defined as a smooth section of the direct image bundle

$$\Lambda_n = p_*((T_V \mathcal{ZF}_g)^{-n}) \rightarrow QF_g.$$

The fibre of  $\Lambda_n$  over  $t \in QF_g$  is the vector space  $B_n(\Omega^t, \Gamma^t)$ . The hyperbolic metric  $\rho^t$  on  $\Gamma^t \backslash \Omega^t$  defines a natural Hermitian metric on the line bundle  $\Lambda_n$  by (2.1).

Analogously, there exist fibre spaces  $p_{\pm} : (\mathcal{QF}_g)_{\pm} \rightarrow QF_g$  with fibre  $\pi^{-1}(t) = \Gamma^t \backslash \Omega_{\pm}^t$  for  $t \in QF_g$ , and consequently we may define families of  $n$ -differentials in  $B_n(\Omega_{\pm}^t, \Gamma^t)$ .

For a harmonic Beltrami differential  $\mu$  in the unit ball of  $\Omega_{-1,1}(\Omega, \Gamma)$ , the pull-back of an  $n$ -differential  $\phi^{\varepsilon} \in B_n(\Omega^{\varepsilon\mu}, \Gamma^{\varepsilon\mu})$  is an  $n$ -differential  $w_{\varepsilon\mu}^*(\phi^{\varepsilon}) \in B_n(\Omega, \Gamma)$  defined by

$$w_{\varepsilon\mu}^*(\phi^{\varepsilon}) = \phi^{\varepsilon} \circ w_{\varepsilon\mu} \left( (w_{\varepsilon\mu})_z \right)^n,$$

where  $w_{\varepsilon\mu} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is the solution of the Beltrami equation corresponding to  $\mu$ . The Lie derivatives of the family  $\phi^{\varepsilon}$  in the directions  $\mu$  and  $\bar{\mu}$  are defined by

$$\begin{aligned} L_{\mu}\phi &= \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} w_{\varepsilon\mu}^*(\phi^{\varepsilon}) \in B_n(\Omega, \Gamma) \\ \text{and } L_{\bar{\mu}}\phi &= \left. \frac{\partial}{\partial \bar{\varepsilon}} \right|_{\varepsilon=0} w_{\varepsilon\mu}^*(\phi^{\varepsilon}) \in B_n(\Omega, \Gamma). \end{aligned}$$

Note that a smooth section  $\phi^t$  of  $(\Lambda_n)_{\pm}$  is holomorphic if and only if  $L_{\bar{\mu}}\phi = 0$  in each Bers coordinate chart. If  $\phi_1^t, \dots, \phi_d^t$  are holomorphic sections which form a basis of  $A_n(\Omega_{\pm}^t, \Gamma^t)$  for each  $t \in QF_g$ , we call them a *holomorphically varying basis* of holomorphic  $n$ -differentials.

**2.5. Inversion on the quasi-Fuchsian deformation space.** There is a canonical inversion  $\iota$  on the quasi-Fuchsian deformation space given by  $\Gamma \mapsto \bar{\Gamma}$ , where  $\bar{\Gamma}$  is the quasi-Fuchsian group

$$(2.7) \quad \bar{\Gamma} = \left\{ \bar{\gamma} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} ; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}.$$

With our conventions, we have  $\Omega_{\pm}(\bar{\Gamma}) = \overline{\Omega_{\mp}(\Gamma)}$ , where  $\overline{\Omega_{\pm}}$  means the set of  $z$  such that  $\bar{z} \in \Omega_{\pm}$ , with the orientation from the complex plane. The hyperbolic metric on  $\bar{\Omega}$  is given by  $\rho_{\bar{\Omega}}(z) = \rho_{\Omega}(\bar{z})$ . If  $\Gamma \backslash \Omega \simeq X_+ \sqcup X_-$ , then  $\bar{\Gamma} \backslash \bar{\Omega} \simeq \bar{X}_- \sqcup \bar{X}_+$ , where  $\bar{X}_{\pm}$  is obtained from  $X_{\pm}$  by taking the conjugate of all local coordinates. Note that as a hyperbolic manifold,  $\bar{X}_{\pm}$  is isometric to  $X_{\pm}$  by an orientation-reversing isometry. The fixed submanifold of the inversion  $\iota$  is precisely the real submanifold of Fuchsian groups. On the Bers coordinate with a Fuchsian basepoint  $\Gamma_0$ , the inversion is realized explicitly by  $\iota(\mu)(z) = \overline{\mu(\bar{z})}$ .

**2.6. Classical Liouville action.** In [TT03a], Takhtajan and the second author constructed the classical Liouville action  $S : QF_g \rightarrow \mathbb{R}$  on the quasi-Fuchsian space, which has the following properties:

- CL1**  $S$  is a real analytic function on  $QF_g$ .
- CL2**  $S$  is invariant under the inversion  $\iota$ , that is,  $S(\Gamma) = S(\bar{\Gamma})$ .
- CL3** Restricted to the real submanifold of Fuchsian groups,  $S$  is a constant, equal to  $8\pi(2g - 2)$ .
- CL4**  $-S$  is a potential for the Weil-Petersson metric on the quasi-Fuchsian deformation space. Namely, at a point  $\Gamma$ , for all  $\mu, \nu \in \Omega_{-1,1}(\Gamma)$ ,

$$\partial_{\mu} \partial_{\bar{\nu}}(-S) = \langle \mu, \nu \rangle.$$

**2.7. Selberg Zeta function and D'Hoker-Phong theorem.** The Selberg zeta function  $Z(s)$  of a Riemann surface  $X \simeq \Gamma \backslash \mathbb{H}_+$ , where  $\Gamma$  is a Fuchsian group, is defined for  $\operatorname{Re} s > 1$  by the absolutely convergent product

$$Z(s) = \prod_{\{\gamma_0\}} \prod_{m=0}^{\infty} (1 - \lambda(\gamma_0)^{s+m}),$$

where  $\gamma_0$  runs over the set of conjugacy classes of primitive hyperbolic elements of  $\Gamma$ , and  $0 < \lambda(\gamma) < 1$  is the multiplier of  $\gamma$ . The function  $Z(s)$  has a meromorphic continuation to the whole  $s$ -plane. In [DP86], D'Hoker and Phong proved that for  $n \geq 2$ ,

$$(2.8) \quad \det \Delta_n = c_{g,n} Z(n).$$

Here  $c_{g,n}$  is an explicitly known positive constant depending only on  $g$  and  $n$ , and not on  $X$  (we refer the reader to [DP86] for the precise expression for  $c_{g,n}$ ).

### 3. GENERALIZED LOCAL INDEX THEOREM ON THE QUASI-FUCHSIAN SPACE

In [ZT87a] (see also [TZ91]), Takhtajan and Zograf proved the Local Index Theorem:

**Theorem.** *Let  $\Gamma$  be a marked Fuchsian group corresponding to a point in the Teichmüller space  $T_g$ , with  $\Gamma \backslash \mathbb{H}_+ \simeq X$  a compact Riemann surface of genus  $g > 1$ . Write  $[X]$  for the corresponding marked Riemann surface. Then we have, for all  $\mu, \nu \in \Omega_{-1,1}(\mathbb{H}_+, \Gamma)$ ,*

$$\partial_\mu \partial_{\bar{\nu}} \log \frac{\det \Delta_n(X)}{\det N_n([X])} = \frac{6n^2 - 6n + 1}{12\pi} \langle \mu, \nu \rangle,$$

where  $(N_n([X]))_{jk} = \langle \phi_j, \phi_k \rangle$  is the period matrix of a holomorphically varying basis  $\phi_1, \dots, \phi_d$  of  $A_n(X)$ , i. e. of holomorphic  $n$ -differentials of the Riemann surface  $X$ .

Note that we write  $N_n([X])$  since we may find a holomorphically varying basis globally only over  $T_g$ , not over the moduli space of surfaces  $X$ .

Given  $\Gamma$  a quasi-Fuchsian group with  $\Gamma \backslash \Omega \simeq X \simeq X_+ \sqcup X_-$  a union of compact Riemann surfaces  $X_\pm$  of genus  $g > 1$ , the determinant of the Laplacian acting on  $n$ -differentials is the product

$$\det \Delta_n(X) = \det \Delta_n(X_+) \det \Delta_n(X_-) = c_{g,n}^2 Z_{X_+}(n) Z_{X_-}(n),$$

and, given choices of holomorphically varying bases of holomorphic  $n$ -differentials on  $X_+$  and  $X_-$ , the period matrix  $N_n([X])$  is defined as

$$\det N_n([X]) = \det N_n([X_+]) \det N_n([X_-]).$$

It is straightforward to generalize the Local Index Theorem to quasi-Fuchsian groups:

**Theorem.** *Let  $\Gamma$  be a marked quasi-Fuchsian group corresponding to a point in the quasi-Fuchsian space  $QF_g$ , with  $\Gamma \backslash \Omega \simeq X \simeq X_+ \sqcup X_-$  a union of two compact Riemann surfaces of genus  $g > 1$ . Write  $[X_\pm]$  for the corresponding marked Riemann surfaces. Then we have, for all  $\mu, \nu \in \Omega_{-1,1}(\Omega, \Gamma)$ ,*

$$\partial_\mu \partial_{\bar{\nu}} \log \frac{\det \Delta_n(X)}{\det N_n([X])} = \partial_\mu \partial_{\bar{\nu}} \log \frac{\det \Delta_n(X_+)}{\det N_n([X_+])} \frac{\det \Delta_n(X_-)}{\det N_n([X_-])} = \frac{6n^2 - 6n + 1}{12\pi} \langle \mu, \nu \rangle.$$



We call this the *Generalized Local Index Theorem*.

*Proof.* Simply use the isomorphism  $QF_g \simeq T_g \times T_g$  to split the tangent space  $T_t QF_g \simeq T_t T_g \oplus T_t T_g$  at each point  $t \in QF_g$ .  $\square$

#### 4. THE FUNCTION $F(n)$

Analogous to the Selberg zeta function  $Z(s)$ , given a quasi-Fuchsian group  $\Gamma$  and an integer  $n \geq 2$ , we define the function

$$F(n) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - \lambda(\gamma_0)^{n+m}),$$

where  $\{\gamma\}$  runs over the set of conjugacy classes of primitive elements of  $\Gamma$ , omitting the identity, and  $|\lambda(\gamma)| < 1$  the multiplier of  $\gamma$ . The product converges absolutely if and only if the series  $\sum_{\{\gamma\}} \sum_{m=0}^{\infty} |\lambda_\gamma|^{m+n}$  converges. It is straightforward to prove that this series converges provided that the multiplier series  $\sum_{[\gamma]} |\lambda_\gamma|^n$  converges, where  $[\gamma]$  runs over all distinct conjugacy classes (not necessarily primitive) in  $\Gamma$ . The argument of Buser for Schottky groups [Büs96] goes through for quasi-Fuchsian groups, showing that multiplier series converges if the Poincaré series  $\sum_{[\gamma]} |\gamma'(z)|^n$  converges, and it is a classical fact that this converges when  $n \geq 2$ .

**Lemma 4.1.**

- (i)  $F(n)$  is a holomorphic function on the quasi-Fuchsian deformation space.
- (ii) Under the inversion  $\iota$  on the quasi-Fuchsian deformation space, the function  $F(n)$  transforms as  $F(n)(\bar{\Gamma}) = \overline{F(n)(\Gamma)}$ .
- (iii) Restricted to the real submanifold of Fuchsian groups,  $F(n)$  is real and coincides with  $Z(n)$ .

*Proof.* Properties (ii) and (iii) are immediate from the definitions. It is easy to show that the multiplier  $\lambda(\gamma)$  is a holomorphic function of the entries of  $\gamma$ , whenever  $0 < \lambda(\gamma) < 1$ . Combining this with (2.6) establishes property (i).  $\square$

These properties characterize  $F(n)$  uniquely; see Lemma 6.1.

#### 5. BERS INTEGRAL OPERATOR AND BERS DUAL BASES

Let  $\Gamma$  be a quasi-Fuchsian group of genus  $g > 1$  with ordinary set  $\Omega_+ \sqcup \Omega_-$  (recall our conventions on quasi-Fuchsian groups from Section 2.1), and let  $n \geq 2$  be an integer. Recall that  $A_n(\Omega_\pm, \Gamma)$ , the space of holomorphic  $n$ -differentials of  $\Gamma$  with support on  $\Omega_\pm$ , is a complex vector space of dimension  $d = (2n-1)(g-1)$ , with a canonical inner product given by

$$\langle \phi, \psi \rangle = \int_{\Gamma \setminus \Omega_\pm} \phi(z) \overline{\psi(z)} \rho(z)^{1-n} d^2 z.$$

**5.1. Bers integral operator.** In [Ber66], Bers introduced complex linear operators

$$K_\pm(\Gamma) : \overline{A_n(\Omega_\pm, \Gamma)} \rightarrow A_n(\Omega_\mp, \Gamma)$$

(we suppress the  $n$  to simplify notation). The two operators  $K_{\pm}(\Gamma)$  (or simply  $K_{\pm}$ ) are defined for  $\phi \in A_n(\Omega_{\pm}, \Gamma)$  and  $z \in \Omega_{\mp}$  by

$$(K_{\pm}\bar{\phi})(z) = c_n \int_{\Omega_{\pm}} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w = \int_{\Gamma \setminus \Omega_{\pm}} K_{\pm}(z, w) \overline{\phi(w)}\rho(w)^{1-n} d^2w,$$

where  $c_n$  is the constant

$$c_n = \frac{2^{2n-2}(2n-1)}{\pi},$$

and, for  $z \in \Omega_{\mp}$ ,  $w \in \Omega_{\pm}$ ,

$$K_{\pm}(z, w) = c_n \sum_{\gamma \in \Gamma} \frac{\gamma'(z)^n}{(\gamma z - w)^{2n}}.$$

We can also define these operators for the conjugate group  $\bar{\Gamma}$ ; we have

$$K_{\pm}(\bar{\Gamma}) : \overline{A_n(\Omega_{\mp}, \bar{\Gamma})} \rightarrow A_n(\Omega_{\pm}, \bar{\Gamma}).$$

*Remark 5.1.* In [Ber66], the operator  $K_-$  is defined on the Banach space  $\overline{\mathcal{B}_n(\Omega_-, \Gamma)}$ , where  $\mathcal{B}_n(\Omega_-, \Gamma)$  is the space of bounded  $n$ -differentials of  $\Gamma$  with support on  $\Omega_-$ . There is a canonical decomposition

$$\mathcal{B}_n = A_n \oplus \mathcal{N}_n,$$

where  $\mathcal{N}_n$  is the subspace of  $\mathcal{B}_n$  consisting of all  $\eta$  such that

$$\int_{\Gamma \setminus \Omega_-} \eta \bar{\phi} \rho^{1-n} = 0,$$

for all holomorphic  $n$ -differentials  $\phi$  which are  $L^1$  integrable. Using this characterization of  $\mathcal{N}_n$ , it is easy to see that  $\overline{\mathcal{N}_n}$  lies in the kernel of  $K_-$ . Hence here we define  $K_-$  on the quotient space  $\overline{\mathcal{B}_n/\mathcal{N}_n} \simeq \overline{A_n}$  (and similarly for  $K_+$ ).

Define operators

$$\begin{aligned} \iota_{\pm} : A_n(\Omega_{\pm}, \Gamma) &\rightarrow A_n(\overline{\Omega_{\pm}}, \bar{\Gamma}), \\ \bar{\iota}_{\pm} : \overline{A_n(\Omega_{\pm}, \Gamma)} &\rightarrow \overline{A_n(\overline{\Omega_{\pm}}, \bar{\Gamma})} \end{aligned}$$

by  $(\iota_{\pm}\phi)(z) = \overline{\phi(\bar{z})}$  and  $(\bar{\iota}_{\pm}\bar{\phi})(z) = \phi(\bar{z})$  for each  $\phi \in A_n(\Omega_{\pm}, \Gamma)$  and  $z \in \overline{\Omega_{\pm}}$ .

**Lemma 5.2.**

- (i)  $K_{\pm}^*\phi = \overline{K_{\mp}\bar{\phi}}$  for all  $\phi$  in  $A_n(\Omega_{\mp}, \Gamma)$ .
- (ii)  $\iota_{\mp}K_{\pm}(\Gamma) = K_{\mp}(\bar{\Gamma})\bar{\iota}_{\pm}$  and  $\bar{\iota}_{\pm}K_{\pm}^*(\Gamma) = K_{\mp}^*(\bar{\Gamma})\iota_{\mp}$ .
- (iii) When  $\Gamma$  is Fuchsian,  $K_{\pm}\bar{\phi} = \iota_{\pm}\phi$  for all  $\phi$  in  $A_n(\Omega_{\pm}, \Gamma)$ .

*Proof.* For  $z \in \Omega_{\pm}$  and  $\phi \in A_n(\Omega_{\mp}, \Gamma)$ , the adjoint operator  $K_{\pm}^* : A_n(\Omega_{\mp}, \Gamma) \rightarrow \overline{A_n(\Omega_{\pm}, \Gamma)}$  is given by

$$(K_{\pm}^*\phi)(z) = c_n \int_{\Omega_{\mp}} \frac{\phi(w)\rho(w)^{1-n}}{(\bar{z}-\bar{w})^{2n}} d^2w,$$

hence

$$\overline{K_{\pm}^*\phi(z)} = c_n \int_{\Omega_{\mp}} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w = (K_{\mp}\bar{\phi})(z),$$

establishing (i). For  $z \in \overline{\Omega_{\mp}}$  and  $\phi \in A_n(\Omega_{\pm}, \Gamma)$ ,

$$\begin{aligned} (\iota_{\mp} K_{\pm}(\Gamma) \overline{\phi})(z) &= \overline{(K_{\pm}(\Gamma) \overline{\phi})(\overline{z})} = c_n \int_{\Omega_{\pm}} \frac{\phi(w) \rho(w)^{1-n}}{(z - \overline{w})^{2n}} d^2 w \\ &= c_n \int_{\Omega_{\pm}} \frac{\phi(\overline{w}) \rho(\overline{w})^{1-n}}{(z - w)^{2n}} d^2 w \\ &= (K_{\mp}(\overline{\Gamma}) \iota_{\pm} \overline{\phi})(z), \end{aligned}$$

which proves the first part of (ii); the second part is similar. When  $\Gamma$  is Fuchsian, property (iii) is the reproducing formula — see [Ber66].  $\square$

Bers proved that the operator  $K_{\pm}$  is invertible. Hence  $\kappa_{\pm} = K_{\pm} K_{\pm}^*$  is a self adjoint positive definite operator.

**Lemma 5.3.**

- (i)  $\kappa_{+}$  and  $\kappa_{-}$  have the same eigenvalues.
- (ii)  $\kappa_{\pm}(\overline{\Gamma})$  and  $\kappa_{\mp}(\Gamma)$  have the same eigenvalues.
- (iii) If  $\Gamma$  is Fuchsian,  $\kappa_{\pm}(\Gamma)$  is the identity.

Note that properties (i) and (ii) imply that  $\kappa_{\pm}(\Gamma)$  and  $\kappa_{\pm}(\overline{\Gamma})$  have the same eigenvalues.

*Proof.* The map  $\phi \mapsto K_{+} \overline{\phi}$  from  $A_n(\Omega_{+}, \Gamma) \rightarrow A_n(\Omega_{-}, \Gamma)$  conjugates  $\kappa_{+}$  with  $\kappa_{-}$ :

$$K_{+} K_{+}^* (K_{+} \overline{\phi}) = K_{+} K_{+}^* \overline{K_{-}^* \phi} = K_{+} \overline{K_{-} K_{-}^* \phi},$$

which establishes (i). To prove (ii), note that the map  $\iota_{-}$  conjugates  $\kappa_{+}(\overline{\Gamma})$  with  $\kappa_{-}(\Gamma)$ :

$$\iota_{-} K_{+}(\Gamma) K_{+}^*(\Gamma) = K_{-}(\overline{\Gamma}) \iota_{+} K_{+}^*(\Gamma) = K_{-}(\overline{\Gamma}) K_{-}^*(\overline{\Gamma}) \iota_{-}.$$

Property (iii) follows from part (iii) of the previous lemma.  $\square$

**5.2. Bers dual bases.** We choose a basis  $\phi_1^+, \dots, \phi_d^+$  of  $A_n(\Omega_{+}, \Gamma)$  and expand the kernel  $K_{-}(z, w)$  with respect to this basis by

$$K_{-}(z, w) = \sum_{k=1}^d \phi_k^+(z) \phi_k^-(w).$$

We define the period matrices  $N_{\pm} = N_n([X_{\pm}])$  by

$$(N_{\pm})_{kl} = \langle \phi_k^{\pm}, \phi_l^{\pm} \rangle.$$

Then

$$\kappa_{-}(z, w) = \int_{\Gamma \setminus \Omega_{-}} K_{-}(z, u) \overline{K_{-}(w, u)} \rho(u)^{1-n} d^2 u = \sum_{j=1}^d \sum_{k=1}^d \phi_j^+(z) (N_{-})_{jk} \overline{\phi_k^+(w)},$$

and hence

$$(\kappa_{-} \phi_l^+)(z) = \sum_{j=1}^d \left( \sum_{k=1}^d (N_{+})_{lk} (N_{-})_{jk} \right) \phi_j^+(z).$$

Namely, with respect to the basis  $\phi_1^+, \dots, \phi_d^+$ , the matrix for the operator  $\kappa_{-}$  is given by  $N_{+} N_{-}^T$ . The invertibility of  $\kappa_{-}$  then shows that  $\phi_1^-, \dots, \phi_d^-$  is a basis of  $A_n(\Omega_{-}, \Gamma)$ , and we say that this basis is *Bers dual* to  $\phi_1^+, \dots, \phi_d^+$ . Note that we have  $\det \kappa_{-} = \det N_{+} \det N_{-}$ .

**5.3. Properties of period matrices.** Suppose that  $\phi_1^+, \dots, \phi_d^+$  is a basis for  $A_n(\Omega_+, \Gamma)$  chosen globally on  $QF_g$  and varying holomorphically, and  $\phi_1^-, \dots, \phi_d^-$  is a Bers dual basis for  $A_n(\Omega_-, \Gamma)$ . Let  $N_\pm$  be the corresponding period matrices.

**Lemma 5.4.**

- (i) *The basis  $\phi_1^-, \dots, \phi_d^-$  varies holomorphically on  $QF_g$ .*
- (ii)  *$\det N_+ \det N_-$  is invariant under the inversion  $\iota$  on  $QF_g$ .*
- (iii)  *$\det N_+ \det N_- = 1$  on the real submanifold of Fuchsian groups.*

*Proof.* If  $\mu \in \Omega_{-1,1}(\Gamma)$  is a harmonic Beltrami differential, and  $w_{\varepsilon\mu}$  is the q. c. mapping defined by  $\varepsilon\mu$ , then by (2.4) and (2.5), we have

$$\left. \frac{\partial}{\partial \bar{\varepsilon}} \right|_{\varepsilon=0} ((w_{\varepsilon\mu} \boxtimes w_{\varepsilon\mu})^* K_-)(z, w) = 0,$$

the symbol  $\boxtimes$  indicating that we are pulling  $K_-(z, w)$  back as an  $n$ -differential in each variable. Hence,

$$\sum_{k=1}^d \left( \left. \frac{\partial}{\partial \bar{\varepsilon}} \right|_{\varepsilon=0} w_{\varepsilon\mu}^* (\phi_k^-)^{\varepsilon\mu} \right) (w) (\phi_k^+)^{\varepsilon\mu}(z) + (\phi_k^-)^{\varepsilon\mu}(w) \left( \left. \frac{\partial}{\partial \bar{\varepsilon}} \right|_{\varepsilon=0} w_{\varepsilon\mu}^* (\phi_k^+)^{\varepsilon\mu} \right) (z) = 0.$$

Consequently, if the basis  $\phi_1^+, \dots, \phi_d^+$  varies holomorphically with respect to moduli, i. e. for all  $k$ ,

$$\left( \left. \frac{\partial}{\partial \bar{\varepsilon}} \right|_{\varepsilon=0} w_{\varepsilon\mu}^* (\phi_k^+)^{\varepsilon\mu} \right) (z) = 0,$$

then

$$\left( \left. \frac{\partial}{\partial \bar{\varepsilon}} \right|_{\varepsilon=0} w_{\varepsilon\mu}^* (\phi_k^-)^{\varepsilon\mu} \right) (z) = 0$$

for all  $k$  as well, which proves (i).

Properties (ii) and (iii) follows from Lemma 5.3, since  $\det \kappa_- = \det N_+ \det N_-$ .  $\square$

**5.4. Global choice of basis over  $QF_g$ .** We may choose a holomorphically varying basis of  $n$ -differentials globally over the Teichmüller space by using the Bers embedding. Fix a marked Riemann surface  $[X_+]$ , and choose a fixed basis  $\phi_1^+, \dots, \phi_d^+$  of  $A_n(X_+)$ . Then for any  $[X_-]$ , take the corresponding quasi-Fuchsian group, identify  $\phi_1^+, \dots, \phi_d^+$  with a basis of  $A_n(\Omega_+, \Gamma)$ , and let  $\phi_1^-, \dots, \phi_d^-$  be the Bers dual basis of  $A_n(\Omega_-, \Gamma)$ . By Lemma 5.4, this basis varies holomorphically. (This is essentially equivalent to a construction by Bers; see [Ber81] and references therein.)

Now, assume a choice of holomorphically varying basis of  $n$ -differentials  $\phi_1^+, \dots, \phi_d^+$  of  $A_n(X_+)$  has been made at each point  $[X_+]$  in the Teichmüller space. Given any  $[X_-]$ , take the corresponding quasi-Fuchsian group, identify  $\phi_1^+, \dots, \phi_d^+$  with a basis of  $A_n(\Omega_+, \Gamma)$ , and let  $\phi_1^-, \dots, \phi_d^-$  be the Bers dual basis of  $A_n(\Omega_-, \Gamma)$ . By Lemma 5.4, this basis varies holomorphically.

Consequently we have shown that it is possible to make a choice of holomorphically varying bases  $\phi_1^\pm, \dots, \phi_d^\pm$  of  $A_n(\Omega_\pm, \Gamma)$  globally over  $QF_g$ , in such a way that  $\phi_1^-, \dots, \phi_d^-$  is Bers dual to  $\phi_1^+, \dots, \phi_d^+$  at each point of  $QF_g$ .

## 6. HOLOMORPHIC FACTORIZATION OF DETERMINANTS OF LAPLACIANS

In this section, we prove our main theorem:

**Theorem.** *Let  $\Gamma$  be a quasi-Fuchsian group simultaneously uniformizing compact Riemann surfaces  $X_+$  and  $X_-$  of genus  $g > 1$ . Then for  $n \geq 2$ ,*

$$\frac{\det \Delta_n(X_+)}{\det N_n([X_+])} \frac{\det \Delta_n(X_-)}{\det N_n([X_-])} = a_{g,n} |F_\Gamma(n)|^2 \exp \left( -\frac{6n^2 - 6n + 1}{12\pi} S_\Gamma \right).$$

Here the Laplacian is computed in the hyperbolic metric;  $N_n([X_\pm])$  are period matrices of bases of  $\ker \Delta_n(X_\pm)$ , Bers dual in the sense of Section 5.2 and chosen globally over the quasi-Fuchsian space; and  $S_\Gamma$  is the classical Liouville action defined in Section 2.6. The function

$$F_\Gamma(n) = \prod_{\{\gamma\} \in \Gamma} \prod_{m=0}^{\infty} (1 - \lambda(\gamma)^{n+m})$$

is defined in more detail in Section 4; and  $a_{g,n}$  is the positive constant

$$a_{g,n} = c_{g,n}^2 \exp \left( \frac{(6n^2 - 6n + 1)(4g - 4)}{3} \right),$$

where  $c_{g,n}$  is the constant from the D'Hoker-Phong formula (2.8).

First, we prove a ‘‘Schwarz reflection’’ lemma for pluriharmonic functions:

**Lemma 6.1.** *Let  $V$  be an open, convex subset of  $\mathbb{C}^m$ , such that  $V \cap \mathbb{R}^m$  is nonempty. Suppose that  $h : V \rightarrow \mathbb{R}$  satisfies the following conditions:*

- (i)  $\frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} = 0$  everywhere in  $V$ , for all  $j, k \in \{1, \dots, m\}$ .
- (ii)  $h(x_1, \dots, x_m) = 0$  for all  $(x_1, \dots, x_m) \in V \cap \mathbb{R}^m$ .
- (iii)  $h(\bar{z}_1, \dots, \bar{z}_m) = h(z_1, \dots, z_m)$  for all  $(z_1, \dots, z_m) \in V$  such that  $(\bar{z}_1, \dots, \bar{z}_m) \in V$ .

Then  $h$  is identically zero on  $V$ .

*Proof.* Fix  $a, b \in \mathbb{R}^m$ , and let  $W = \{z \in \mathbb{C} : az + b \in V\}$ . Define  $f : W \rightarrow \mathbb{R}$  by  $f(z) = h(az + b)$ . Property (i) implies that  $f$  is harmonic on  $W$ . Property (ii) implies that  $f(x) = 0$  for all  $x \in W \cap \mathbb{R}$ . Hence by the Schwarz reflection principle, we have  $f(\bar{z}) = -f(z)$  for all  $z$  in some neighbourhood of  $W \cap \mathbb{R}$ . On the other hand,  $f(\bar{z}) = f(z)$  whenever  $z, \bar{z} \in W$  by (iii), so  $f(z) = 0$  for all  $z$  in some neighbourhood of  $W \cap \mathbb{R}$ . But since  $V$  is convex,  $W$  is connected, so  $f(z) = 0$  for all  $z \in W$ .

Hence  $h$  is zero at all points in  $V$  of the form  $az + b$  for some  $a, b \in \mathbb{R}^m$  and some  $z \in \mathbb{C}$ . But it is easy to check that every point in  $\mathbb{C}^m$  is of this form, so  $h$  is identically zero on  $V$ .  $\square$

Now we return to the proof of the theorem. Let

$$h = \log \left( \frac{\det \Delta_n(X_+)}{\det N_n([X_+])} \frac{\det \Delta_n(X_-)}{\det N_n([X_-])} \right) / a_{g,n} |F_\Gamma(n)|^2 \exp \left( -\frac{6n^2 - 6n + 1}{12\pi} S_\Gamma \right).$$

The function  $h$  is real-valued on the quasi-Fuchsian deformation space  $QF_g$ . We claim that

- (i)  $h$  is pluriharmonic on  $QF_g$
- (ii)  $h$  is invariant under the inversion  $\iota$

(iii)  $h = 0$  on the real submanifold of Fuchsian groups.

Property (i) follows directly from the generalized local index theorem (Section 3) and the fact that  $F_\Gamma(n)$  is holomorphic on  $QF_g$  (Section 4). Property (ii) follows from the transformations under  $\iota$ , established in Sections 2.2, 4, 2.6, and 5.3, of the factors appearing in  $h$ . Using the fact that  $S_\Gamma$  is constant on the real submanifold of Fuchsian groups, property (iii) reduces to the D'Hoker-Phong formula (2.8).

Writing  $h$  in the local coordinates on  $QF_g$  described in Section 2.3, we see that it satisfies the conditions of Lemma 6.1, and hence is identically zero on  $QF_g$ , proving the theorem.

*Remark 6.2.* The theorem should be considered as an equality of functions over the quasi-Fuchsian space  $QF_g$ . However, we can obtain as a corollary an equality of functions over the Teichmüller space  $T_g$ , by means of the Bers embedding. Fix the marked Riemann surface  $[X_-]$  and the basis  $\phi_1^-, \dots, \phi_d^-$  of holomorphic  $n$ -differentials on  $[X_-]$ . Then  $\det \Delta_n(X_-)/\det N_n([X_-])$  is constant, and all other quantities in the theorem are functions only of  $[X_+] \in T_g$ .

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